You are given 4 hours. Each problem is worth 7 points.
Problems are meant to be kept confidential even after the test until 1 April 2024.

Problem 1. *Let $P(x), Q(x)$ and $R(x)$ be non-constant polynomials with positive integer coefficients. Prove that there do not exist any positive integers $a, b, c \geq 2$ such that*

$$^{a}P(x) + {}^{b}Q(x) = {}^{c}R(x)$$

holds for all $x \in \mathbb{N}$ where ^{a}t denotes tetration.

Problem 2. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

1. $f(f(x)) - xf(x) + yf(x) = f(f(y)) + xf(y) - yf(y)$
2. $f(xy) = f(x)f(y)$

Problem 3. *For any even integer $k > 2$, call a subset of $\{1, 2, .., k\}$ "good" if it contains half of those integers and no two of them divide each other. Find the smallest possible number that belongs in a good set, in terms of k.*

Problem 4. *Vanilla and Celestia play a game (which you just lost!) on a lattice polygon P. Firstly, Vanilla marks one cell C as a landmine, known to Celestia, whose task is to put counters on every cell except from C under the following rules:*

- *On the first turn he can put a counter on any cell.*
- *On each turn after that, he can put a counter on a cell if and only if the number of counters in the same column as the cell plus the number of counters in the same row is odd.*

It is given that there exist cells $C_1, C_2, C_3, \ldots$ in distinct rows and columns in the polygon such that Celestia can fill every other cell in the polygon with counters if Vanilla chooses any of these as the landmine. Prove that no matter which cell in the polygon Vanilla marks as a landmine, Celestia can complete his task.

Remark: A cell is a square bounded by 4 adjacent lattice points.

Problem 5. *Given is a triangle ABC with incenter I. Consider S, the intersection of the Euler lines of $\triangle ABC$ and $\triangle ABI$. Denote by D the intersection of CI with AB and E as the intersection of AS with the C-altitude. Prove that DE intersects IS on the B-altitude*

Remark: The Euler line of a triangle is the line going through the orthocenter and centroid.